General Scattering Amplitude

The Formal Solution And Application

Contents

1 Motivation

Recalling the history of the physics, the problem of scattering is encountered in almost every region in physics. For example, the incident electron to the atom, gravitational slingshot, and even in statistical theory we need the scattering to construct non-equibrium theory etc.

However, though we have known the significance of the scattering amplitude, the methods we have learnt are almost all classical. We actually are lacking knowledge how to solve it in quantum.

So, today I'd like to introduce the topic of scattering amplitude in quantum mechanics. And I'd like to talk mostly in formal theory, which means we would mainly focus on the solution's general math structure instead of the exact solution under certain circumstances.

You can get a draft of what will going to be talked from this contents. In the application section, we will give some result that can be deduced out from the general form of scattering amplitude.

So now, lets begin.

2 Scattering Amplitude In Quantum Mechanics

First, let's look into the problem of Scattering from the classical sense. We will go deeper into the quantum description later. We present the problem definition here.

Theorem 2.1

Consider an particle incident from the infinite distance on the left. It has energy E and point to the right, and for the central is at a distance of b. We would like to know the exit angle $\theta(b, E)$ as a function of b and E

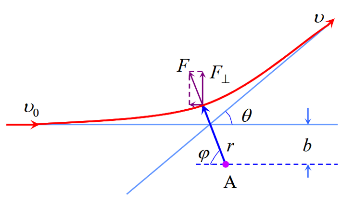

Fig. 1. RutherfordScattering

Under the frame of classical Mechanics, the scattering problem is abstracted into the problem of central force. So we can solve the scattering problem within the Conservation of Angular Momentum.

Theorem 2.2

The problem of central force can be described into solving:

$$\begin{cases} L = mr^2 \frac{d\phi}{dt} = mv_0 b \\ v_\perp = v_0 \sin\theta \end{cases}$$

We can find the exit angle is now actually an ODE problem:

$$\begin{aligned} \mathbf{d}v_\perp = & \frac{F_\perp}{m}\mathbf{d}t \\ v_0 \mathbf{d}\sin\theta = & \frac{F\sin\phi}{m}\mathbf{d}t \end{aligned}$$

We describe the classical picture and the result. However, what about in the quantum mechanics? Maybe someone will say :" How about just correspond the problem definition direct into the quantum mechanics?" Alright, that seems great. But we will experience great challenge at the particle description, In quantum mechanics the particle is actually described in wave packet. Though we can just solve out the Schrodinger Equation and find all the eigenfunction to describe the particle, but maybe we can think a little bit more into the essence of the scattering problem.

While we are doing scattering problem in classical, we are talking about the particle position for the sake that position is one of the limited variable that describe the state. Meanwhile, we are discussing only the exit angle because it's an one-to-one correspondence in the classical situation. From the above description, you may generalize and say: "Hey, the scattering problem is eventually finding some mapping from one state to the others!" Yeah, this is an excellent generalization of the classical scattering problem in quantum way.

So, from the discussion above, what we will actually calculate in quantum mechanics is the amplitude from ψ_α to ψ_β, This is why we call it scattering amplitude. We give the problem definition here:

Theorem 2.3

For a physical system with $V \longrightarrow 0$ at infinite distance:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{r})$$

We would like to know the amplitude from ψ_α to ψ_β

Let's go into the next section.

3 S-Matrix

At the first sight of the word "S-Matrix", you might find it a bit confusing, what is the S mean here, where does this matrix origin? Actually, here you should see the letter "S" as the abbreviation of "scattering", and the matrix as the corelation between different state.

3.1 In and Out State

While we are treating interaction between particles, we actually want to know the scattering between the states right before the interaction and right after the interaction as:

$$\hat{H}\psi_\alpha = E_\alpha \psi_\alpha^\pm$$

These two states are always called In and Out states. But we will encounter difficulty solving out the exact expression at this two point. So finding a way to describe this two states should be proposed. A good way to do this is implied by the approximated-free at infinite distance, because the Scattering Problem has the property that $V \longrightarrow 0$ at infinite.

So from the free property at infinite, we can use free hamiltonian $\hat{H}_0$ to describe the state at $t \longrightarrow \pm\infty$. And by using the system hamiltonian $\hat{H} = \hat{H}_0 + V$ to do time-evolution back to the In and Out State point.

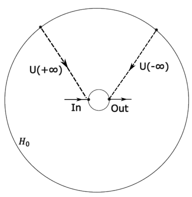

Fig. 2. Some Intuitive Picture

So now, what we have to deal is just to find the free-state on the infinite distance. And this can be done by just reversal evolve the free-state in the bulk. Let $\phi_\alpha^\pm$ indicates the state on the infinite boundary corresponding $\psi_\alpha^\pm$, so we will have an equation (Comes from the last paragraph's statement, we are just doing something like time evolution or reference frame transformation, each can be explained under passive or active view on transformation):

$$\begin{aligned} &\hat{H}_0\phi_\alpha = E_\alpha\phi_\alpha \\ &\psi_\alpha^\pm = \Omega(\mp\infty)\phi_\alpha, \quad \Omega(\tau) = e^{iH\tau}e^{-iH_0\tau} \end{aligned} \tag{1}$$

After we give out the definition of In and Out state, we can finally define the S-Matrix:

Theorem 3.1

For each in state ψ_α^+ and out state ψ_β^-, there exist a S-Matrix element $S_{\alpha\beta}$:

$$S_{\alpha\beta} = \left\langle \psi_\beta^- | \psi_\alpha^+ \right\rangle$$

And also from 1, we know the operator expression of $\hat{\mathrm{S}}$:

$$\hat{\mathrm{S}} = \Omega(+\infty)\Omega^\dagger(-\infty) = U(+\infty, -\infty) \tag{2}$$

Where $U(\tau, \tau_0) \equiv \exp(iH_0\tau)\exp(-iH(\tau-\tau_0))\exp(-iH_0\tau_0)$

3.2 Lippmann-Schwinger Equation

Though we can just use the 2 to give some further application, but some detailed general math structure is already hidden in the matrix element, we would like to find it out. But to avoid using the explicit expression of the $\hat{H}$, we would like to introduce the Lippmann-Schwinger Equation to see some conservation law in the matrix element.

Theorem 3.2

For hamiltonian like $H = H_0 + V$, if we have :

$$\begin{aligned} &H_0 \left|\phi\right\rangle = E\left|\phi\right\rangle \\ &(H_0 + V)\left|\psi\right\rangle = E\left|\psi\right\rangle \end{aligned}$$

Then we have an iterative solution:

$$|\psi^{\pm}\rangle = |\phi\rangle + \frac{1}{E - H_0 \pm i\epsilon} V |\psi^{\pm}\rangle \tag{3}$$

From the above subsection, we know the In and Out states at $t \to \pm\infty$ have the properties of:

$$\psi_\alpha^\pm \to \phi_\alpha$$

Meanwhile, from the definition of S-matrix element 2, we also have:

$$\psi_\beta^\pm = \int \mathbf{d}\alpha S_{\beta\alpha} \psi_\alpha^\pm \tag{4}$$

So from the above two equation, we actually can derive:

$$\psi_\beta^\pm = \int \mathbf{d}\alpha \phi_\alpha \tag{5}$$

Now just take the Lippmann-Schwinger Equation 3 into 6, then the structure of S-Matrix element is:

$$S_{\alpha\beta} = \delta(\beta - \alpha) - 2\pi i \delta(E_\alpha - E_\beta) T_{\beta\alpha} \tag{6}$$

where $T_{\beta\alpha} = \left\langle \phi_\beta \middle| V \middle| \psi_\alpha^+ \right\rangle$. Now let's explain what does the matrix element means, the first term tells us the behavior of free state, and the second term tells us only the scattering state follows the energy conservation law can be accepted.

3.3 Formal Solution of S-Matrix

So, is free approximate state at infinite distance the only constrains we can use? Actually no, there are still constrains from various symmetry law. Like space translational invariance, if we apply this symmetry, we could get a much more useful form of S-Matrix:

Theorem 3.3

The S-Matrix element with translational invariance:

$$S_{\beta\alpha} - \delta(\alpha - \beta) = \delta(E_\alpha - E_\beta)\delta^3(P_\alpha - P_\beta) M_{\beta\alpha} \tag{7}$$

Here we minus the $\delta(\alpha - \beta)$ to leave out the non-interaction state, and the $M_{\beta\alpha}$ is smooth function of momentum.

4 Application

So, now we can give some useful application of the Formal Solution of the scattering amplitude. And if time accepts, we can take a glance at the graph method.

4.1 Transition Probability

4.1.1 Box-Method

The transition probability has great significant role in particle experiment, as described in the motivation part. So the relation between the scattering amplitude and the scattering cross-section must be discussed.

However, during our discussion of the amplitude, we encountered the factor $\delta^4(p_\alpha - p_\beta)$. In general integral, it can be explained by a momentum selector, but in discussion of the observable, the square of the delta function must be understand, so how we understand the square part?

The solution is given by the Box Method. What makes us confusing about the square delta function is because the well defined function of a selector in integral is not established in the full euclidean space, so we can't tell the math structure of the delta function. However, if we consider space with finite volume, we could get the selector function's math structure, and next we generalize to infinite volume, we could keep the math structure.

The idea of box method is imagining the system is contained in a Volume V box, and interaction happen during time T, this tells us the relation between the matrix-element in box and the general:

$$S^{Box}_{\beta\alpha} = \left(\frac{(2\pi\hbar)^3}{V}\right)^{(N_\alpha+N_\beta)/2} S_{\beta\alpha} \tag{8}$$

Here N_α and N_β means the particle number in the in and out state. And now we can consider the transition probability:

$$\begin{aligned}\Gamma(\alpha \to \beta) = & \tfrac{P(\alpha\to\beta)}{T} \\ \propto & V^{1-N_\alpha-N_\beta}|M_{\beta\alpha}|^2\end{aligned} \tag{9}$$

And also in experiment, what we measure is always the transition probability into several states collection, so we'd better also give the transition probability into $\mathbf{d}\beta$:

$$\mathbf{d}\Gamma(\alpha \to \beta) \propto V^{1-N_\alpha}|M_{\beta\alpha}|^2\mathbf{d}\beta \tag{10}$$

4.1.2 $N_\alpha = 1 and 2$ Case

The above result 10 actually has two special situation, which the interfering factor of the decay rate is quite special:

Theorem 4.1

- $N_\alpha = 1$, independent of the circumstance volume
- $N_\alpha = 2$, proportion to the final state particle density

5 Conclusion

From all the discussion above, we have seen the significance of the scattering amplitude. Whats more, as we mentioned in the last part, the scattering amplitude also build a bridge to the QFT. And in currently research, people also try to compute the amplitude under more Symmetry Restriction. The computation result not only give us what the experiment will be, but also inspire some mathmatical region.

6 Reference

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- Weinberg, Lectures on Quantum Mechanics.
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